

EXPLICIT EVALUATIONS OF THE HANKEL DETERMINANTS OF A THUE–MORSE-LIKE SEQUENCE

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ABSTRACT. We obtain the explicit evaluations of the Hankel determinants of the formal power series $\prod_{k \geq 0} (1 + Jx^{3^k})$ where $J = (\sqrt{-3} - 1)/2$, and prove that the sequence of Hankel determinants is an aperiodic automatic sequence taking value in $\{0, \pm 1, \pm J, \pm J^2\}$. This research is essentially inspired by the works about Hankel determinants of Thue–Morse-like sequences by Allouche, Peyrière, Wen and Wen (1998), Bacher (2006) and the first author (2013).

1. INTRODUCTION

In 1998, Allouche, Peyrière, Wen and Wen proved that all the Hankel determinants of the Thue–Morse sequence

$$P_2(x) = \prod_{k \geq 0} (1 - x^{2^k}) \quad (1)$$

are nonzero by using determinant manipulation [1], which consists of proving sixteen recurrent relations between determinants. Recently, the first author derived a short proof of APWW’s result by using the Jacobi continued fraction expansion of the underlying sequence [9]. Moreover, he proved that all the Hankel determinants of the following Thue–Morse-like sequence

$$P_3(x) = \prod_{k \geq 0} (1 - x^{3^k}) \quad (2)$$

are nonzero. Those results about Hankel determinants have been shown to have useful applications in Number Theory for studying the irrationality exponents of automatic numbers (see [7, 8]).

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Notice that the Hankel determinants of $P_2(x)$ and $P_3(x)$ do not have any closed-form expressions. The trick to study such determinants is using the modular arithmetic. Surprisingly enough, Bacher obtained the explicit evaluations of the Hankel determinants of the following Thue–Morse-like sequence

$$P_2(x; I) = \prod_{k \geq 0} (1 + Ix^{2^k}), \quad (3)$$

where I is the imaginary unit [3, 4]. Bacher’s method is based on the *category Rec* introduced by himself. Inspired by the above three results, we derived the explicit evaluations of the Hankel determinants of the following Thue–Morse-like sequence

$$P_3(x; J) = \prod_{k \geq 0} (1 + Jx^{3^k}), \quad (4)$$

where $J = (I\sqrt{3} - 1)/2$. Our proof is based on APWW’s method by using direct determinant manipulations.

Consider the sequence $\mathbf{c} = (c_0, c_1, c_2, \dots)$ defined by the generating function

$$\begin{aligned} P_3(x; J) &= \prod_{k \geq 0} (1 + Jx^{3^k}) = c_0 + c_1x + c_2x^2 + \dots \\ &= 1 + Jx + Jx^3 + J^2x^4 + Jx^9 + J^2x^{10} + J^2x^{12} + x^{13} + Jx^{27} + \dots \end{aligned} \quad (5)$$

It is easy to show that the sequence \mathbf{c} can be characterized by the following recurrence relations

$$c_0 = 1, \quad c_{3n} = c_n, \quad c_{3n+1} = Jc_n, \quad c_{3n+2} = 0, \quad (n \geq 0) \quad (6)$$

or equivalently by the morphism σ over alphabet $\mathcal{A} = \{0, 1, J, J^2\}$ where σ is defined as follow

$$1 \mapsto 1J0, \quad J \mapsto JJ^20, \quad J^2 \mapsto J^210, \quad 0 \mapsto 000.$$

Thus \mathbf{c} is an *automatic sequence*, i.e., it can be generated by a finite automaton (see [2]).

Recall that for each sequence of complex numbers $\mathbf{u} = (u_k)_{k=0,1,\dots}$ the corresponding (p, n) -order *Hankel matrix* $H_n^p(\mathbf{u})$ is given by

$$H_n^p(\mathbf{u}) = \begin{pmatrix} u_p & u_{p+1} & \cdots & u_{p+n-1} \\ u_{p+1} & u_{p+2} & \cdots & u_{p+n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{p+n-1} & u_{p+n} & \cdots & u_{p+2n-2} \end{pmatrix}, \quad (7)$$

where $n \geq 1$ and $p \geq 0$. The *Hankel determinant* $|H_n^p(\mathbf{u})|$ is simply the determinant of the Hankel matrix $H_n^p(\mathbf{u})$. By convention, $|H_0^p(\mathbf{u})| = 1$.

Our main result about the Hankel determinants of \mathbf{c} is stated next.

Theorem 1. *Let $H_n^p := H_n^p(\mathbf{c})$ be the (p, n) -order Hankel matrix of the Thue–Morse-like sequence \mathbf{c} defined in (5). Then, the Hankel determinants $|H_n^0|$ and $|H_n^1|$ are characterized by the following recurrence relations*

$$\begin{cases} |H_0^0| &= 1, \\ |H_1^0| &= 1, \\ |H_{3n}^0| &= |H_n^0|, \\ |H_{3n+1}^0| &= |H_{n+1}^0|, \\ |H_{3n+2}^0| &= -J^2 |H_{n+1}^0| \end{cases} \quad (8)$$

and

$$\begin{cases} |H_0^1| &= 1, \\ |H_{3n}^1| &= |H_n^1|, \\ |H_{3n+1}^1| &= J |H_n^1|, \\ |H_{3n+2}^1| &= J |H_{n+1}^1| \end{cases} \quad (9)$$

for all $n \geq 0$.

The first values of the Hankel determinants $|H_n^0|$ and $|H_n^1|$ are reproduced in the following table.

n	$=$	0	1	2	3	4	5	6	7	8	9	10	11
$ H_n^0 $	$=$	1	1	$-J^2$	1	$-J^2$	J	$-J^2$	1	$-J^2$	1	$-J^2$	J
$ H_n^1 $	$=$	1	J	J^2	J	J^2	1	J^2	1	J^2	J	J^2	1

Consider the sequence $\mathbf{s} = (s_0, s_1, s_2, \dots)$ defined by $s_n = c_n + c_{n+1}$. We have

$$\begin{aligned} S(x) &= \frac{(1+x)P_3(x; J) - 1}{x} = s_0 + s_1x + s_2x^2 + \dots \\ &= -J^2 + Jx + Jx^2 - x^3 + J^2x^4 + Jx^8 - x^9 + J^2x^{10} + \dots \end{aligned} \quad (10)$$

and

$$s_{3n} = -J^2 c_n, \quad s_{3n+1} = J c_n, \quad s_{3n+2} = c_{n+1}. \quad (11)$$

The Hankel matrices of the sequences \mathbf{s} are denoted by Σ_n^p . An easy observation shows that

$$\Sigma_n^p = H_n^p + H_n^{p+1}. \quad (12)$$

Theorem 2. *The Hankel determinants $|\Sigma_n^0|$ and $|\Sigma_n^1|$ are characterized by the following recurrent relations*

$$\begin{cases} |\Sigma_0^0| &= 1, \\ |\Sigma_{3n}^0| &= |\Sigma_n^0|, \\ |\Sigma_{3n+1}^0| &= -J^2 |\Sigma_n^0|, \\ |\Sigma_{3n+2}^0| &= -J^2 |\Sigma_{n+1}^0| \end{cases} \quad (13)$$

and

$$\begin{cases} |\Sigma_0^1| &= 1, \\ |\Sigma_{3n}^1| &= |\Sigma_n^1|, \\ |\Sigma_{3n+1}^1| &= J|\Sigma_n^1|, \\ |\Sigma_{3n+2}^1| &= |\Sigma_n^1| \end{cases} \quad (14)$$

for all $n \geq 0$.

The first values of the Hankel determinants $|\Sigma_n^0|$ and $|\Sigma_n^1|$ are reproduced in the following table.

n	$=$	0	1	2	3	4	5	6	7	8	9	10	11
$ \Sigma_n^0 $	$=$	1	$-J^2$	J	$-J^2$	J	-1	J	-1	J	$-J^2$	J	-1
$ \Sigma_n^1 $	$=$	1	J	1	J	J^2	J	1	J	1	J	J^2	J

Theorem 3. For each $p, n \geq 0$, the Hankel determinants $|H_n^p|$ and $|\Sigma_n^p|$ are equal to $0, \pm 1, \pm J, \pm J^2$.

Let us make further comments about Hankel determinants and automatic sequences. Hankel matrices and Hankel determinants of a sequence are strong connected to the moment problem [12] and to the Padé approximation [5, 6]. In [11], Kamae, Tamura and Wen studied the properties of Hankel determinants for the Fibonacci word and give a quantitative relation between the Hankel determinant and the Padé pair. Later, Tamura [13] generalized the results for a class of special sequences. Allouche, Peyrière, Wen and Wen studied the properties of Hankel determinants $|\mathcal{E}_n^p|$ of the Thue-Morse sequence in [1]. They proved that the Hankel determinants $|\mathcal{E}_n^p|$ modulo 2 recognized as a two-dimensional sequence (or *double sequence*) is 2-automatic.

Theorem 1 implies that the first two columns of the two-dimensional sequence $\{|H_n^p|\}_{n,p \geq 0}$, i.e., $\{|H_n^0|\}_{n \geq 0}$ and $\{|H_n^1|\}_{n \geq 0}$, are 3-automatic sequences. They are obviously *aperiodic*, which are different than the Hankel determinants of the Thue-Morse sequence and of the regular paperfolding sequence studied in [1] and [8, 10] respectively.

In Section 2 we establish a key lemma, namely, Lemma 5, which consists a list of recurrent relations between the determinants $|H_n^p|$ and $|\Sigma_n^p|$. Theorems 1-3 are simple consequences of Lemma 5. The recurrent relations them-self are proved in Sections 3 and 4.

2. THE SUDOKU METHOD

The proofs of Theorems 1-3 are based on the method developed by Allouche, Peyrière, Wen and Wen [1], that could be called *sudoku method*. The sudoku method consists some basic determinant manipulations. Matrices are often split into $3 \times 3 = 9$ small blocks.

For each matrix $M = (m_{i,j})_{i,j=1,2,\dots,n}$ of size $n \times n$, denote by M^t the transpose of M . Let $M^{(i)}$ be the $n \times (n-1)$ -matrix obtained by deleting the i -th column of M , and $M_{(i)}$ be the $(n-1) \times n$ -matrix obtained by deleting the i -th row of M . The determinant of the matrix M is denoted by $|M|$. Also, let $\mathbf{0}_{m,n}$ denote the $m \times n$ zero matrix.

For each $n \geq 1$ let $P(n)$ be the $n \times n$ -matrix defined by

$$P(n) = (e_1, e_4, \dots, e_{3n_1-2}, e_2, e_5, \dots, e_{3n_2-1}, e_3, e_6, \dots, e_{3n_3}), \quad (15)$$

where $n_1 = \lfloor \frac{n+2}{3} \rfloor$, $n_2 = \lfloor \frac{n+1}{3} \rfloor$, $n_3 = \lfloor \frac{n}{3} \rfloor$ and e_j is the j -th unit column vector of order n , i.e., the column vector with 1 as the j -th entry and zeros elsewhere. For simplicity, we write P instead of $P(n)$, when no confusion can occur. Obviously, $|P(n)| = \pm 1$. When consider $P(3n), P(3n+1), P(3n+2)$, the following diagram shows the values of n_1, n_2 and n_3 in these cases:

	n_1	n_2	n_3
$3n$	n	n	n
$3n+1$	$n+1$	n	n
$3n+2$	$n+1$	$n+1$	n

Lemma 4. Let $M = (m_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ -matrix and $P = P(n)$ be the matrix, of the same size as M , defined in (15). Then

$$P^t M P = \begin{pmatrix} (m_{3i-2,3j-2})_{n_1 \times n_1} & (m_{3i-2,3j-1})_{n_1 \times n_2} & (m_{3i-2,3j})_{n_1 \times n_3} \\ (m_{3i-1,3j-2})_{n_2 \times n_1} & (m_{3i-1,3j-1})_{n_2 \times n_2} & (m_{3i-1,3j})_{n_2 \times n_3} \\ (m_{3i,3j-2})_{n_3 \times n_1} & (m_{3i,3j-1})_{n_3 \times n_2} & (m_{3i,3j})_{n_3 \times n_3} \end{pmatrix},$$

where $(m_{3i-2,3j-1})_{s \times t}$ means the matrix $(m_{3i-2,3j-1})_{1 \leq i \leq s, 1 \leq j \leq t}$.

Recall that for each sequence of complex numbers $\mathbf{u} = (u_k)_{k=0,1,\dots}$ the corresponding (p, n) -order Hankel matrix $H_n^p(\mathbf{u})$ is defined by (7). Let $K_n^p = K_n^p(\mathbf{u}) := (u_{p+3(i+j-2)})_{1 \leq i,j \leq n}$. When applying Lemma 4 for $M = H_{3n}^p(\mathbf{u})$, $H_{3n+1}^p(\mathbf{u})$, $H_{3n+2}^p(\mathbf{u})$, we get

$$\begin{aligned} & P^t H_{3n}^p(\mathbf{u}) P \\ &= \begin{pmatrix} (u_{p+3(i+j-2)})_{n \times n} & (u_{p+3(i+j-2)+1})_{n \times n} & (u_{p+3(i+j-2)+2})_{n \times n} \\ (u_{p+3(i+j-2)+1})_{n \times n} & (u_{p+3(i+j-2)+2})_{n \times n} & (u_{p+3(i+j-2)+3})_{n \times n} \\ (u_{p+3(i+j-2)+2})_{n \times n} & (u_{p+3(i+j-2)+3})_{n \times n} & (u_{p+3(i+j-2)+4})_{n \times n} \end{pmatrix} \\ &= \begin{pmatrix} K_n^p & K_n^{p+1} & K_n^{p+2} \\ K_n^{p+1} & K_n^{p+2} & K_n^{p+3} \\ K_n^{p+2} & K_n^{p+3} & K_n^{p+4} \end{pmatrix}, \end{aligned} \quad (16)$$

and

$$P^t H_{3n+1}^p(\mathbf{u})P = \begin{pmatrix} K_{n+1}^p & (K_{n+1}^{p+1})^{(n+1)} & (K_{n+1}^{p+2})^{(n+1)} \\ (K_{n+1}^{p+1})^{(n+1)} & K_n^{p+2} & K_n^{p+3} \\ (K_{n+1}^{p+2})^{(n+1)} & K_n^{p+3} & K_n^{p+4} \end{pmatrix}, \quad (17)$$

$$P^t H_{3n+2}^p(\mathbf{u})P = \begin{pmatrix} K_{n+1}^p & K_{n+1}^{p+1} & (K_{n+1}^{p+2})^{(n+1)} \\ K_{n+1}^{p+1} & K_{n+1}^{p+2} & (K_{n+1}^{p+3})^{(n+1)} \\ (K_{n+1}^{p+2})^{(n+1)} & (K_{n+1}^{p+3})^{(n+1)} & K_n^{p+4} \end{pmatrix}. \quad (18)$$

As shown in Sections 3-4, Formulae (16)-(18) can be used to prove the following recurrent relations between the determinants $|H_n^p|$ and $|\Sigma_n^p|$ when the sequence \mathbf{u} is taken from the sequences \mathbf{c} and \mathbf{s} defined in (6) and (11) respectively. Through these eighteen recurrent formulae, we can evaluate all the Hankel determinants $|H_n^p|$ and $|\Sigma_n^p|$ ($n \geq 1, p \geq 0$). Our key lemma is stated next.

Lemma 5. *For each $p \geq 0$ and $n \geq 1$ we have*

- (L1) $|H_{3n}^{3p}| = (-1)^n |H_n^p| \cdot |H_n^{p+1}| \cdot |\Sigma_n^p|,$
- (L2) $|H_{3n+1}^{3p}| = (-1)^n |H_n^{p+1}| \cdot |H_{n+1}^p| \cdot |\Sigma_n^p|,$
- (L3) $|H_{3n+2}^{3p}| = (-1)^{n+1} J^2 |H_{n+1}^p|^2 \cdot |\Sigma_n^{p+1}|,$
- (L4) $|H_{3n}^{3p+1}| = (-1)^n |H_n^{p+1}|^2 \cdot |\Sigma_n^p|,$
- (L5) $|H_{3n+1}^{3p+1}| = (-1)^n J |H_n^{p+1}| \cdot |H_{n+1}^p| \cdot |\Sigma_n^{p+1}|,$
- (L6) $|H_{3n+2}^{3p+1}| = (-1)^n J |H_{n+1}^p| \cdot |H_{n+1}^p| \cdot |\Sigma_n^{p+1}|,$
- (L7) $|H_{3n}^{3p+2}| = (-1)^n |H_n^{p+1}|^2 \cdot |\Sigma_n^{p+1}|,$
- (L8) $|H_{3n+1}^{3p+2}| = 0,$
- (L9) $|H_{3n+2}^{3p+2}| = (-1)^{n+1} |H_{n+1}^{p+1}|^2 \cdot |\Sigma_n^{p+1}|,$
- (L10) $|\Sigma_{3n}^{3p}| = (-1)^n |\Sigma_n^p|^2 \cdot |H_n^{p+1}|,$
- (L11) $|\Sigma_{3n+1}^{3p}| = (-1)^{n+1} J^2 |H_{n+1}^p| \cdot |\Sigma_n^p| \cdot |\Sigma_n^{p+1}|,$
- (L12) $|\Sigma_{3n+2}^{3p}| = (-1)^{n+1} J^2 |H_{n+1}^p| \cdot |\Sigma_{n+1}^p| \cdot |\Sigma_n^{p+1}|,$
- (L13) $|\Sigma_{3n}^{3p+1}| = (-1)^n |H_n^{p+1}| \cdot |\Sigma_n^p| \cdot |\Sigma_n^{p+1}|,$
- (L14) $|\Sigma_{3n+1}^{3p+1}| = (-1)^n J |H_{n+1}^p| \cdot |\Sigma_n^{p+1}|^2,$
- (L15) $|\Sigma_{3n+2}^{3p+1}| = (-1)^{n+1} |H_{n+1}^{p+1}| \cdot |\Sigma_{n+1}^p| \cdot |\Sigma_n^{p+1}|,$
- (L16) $|\Sigma_{3n}^{3p+2}| = (-1)^n |\Sigma_n^{p+1}|^2 \cdot |H_n^{p+1}|,$
- (L17) $|\Sigma_{3n+1}^{3p+2}| = (-1)^n |\Sigma_n^{p+1}|^2 \cdot |H_{n+1}^{p+1}|,$
- (L18) $|\Sigma_{3n+2}^{3p+2}| = 0.$

Corollary 6. *Let $A_n^p = (-1)^n |H_n^{p+1}| \cdot |\Sigma_n^p|$ and $B_n^p = (-1)^n |H_{n+1}^p| \cdot |\Sigma_n^{p+1}|$. Then,*

- (C1) $A_{3n}^{3p} = (A_n^p)^3,$
- (C2) $A_{3n+1}^{3p} = A_n^p (B_n^p)^2,$
- (C3) $A_{3n+2}^{3p} = A_{n+1}^p (B_n^p)^2,$

$$\begin{aligned}
(C4) \quad & B_{3n}^{3p} = (A_n^p)^2 B_n^p, \\
(C5) \quad & B_{3n+1}^{3p} = (B_n^p)^3, \\
(C6) \quad & B_{3n+2}^{3p} = (A_{n+1}^p)^2 B_n^p.
\end{aligned}$$

Proof. (C1) From Equalities (L4) and (L10) stated in Lemma 5 we have

$$\begin{aligned}
A_{3n}^{3p} &= (-1)^{3n} |H_{3n}^{3p+1}| \cdot |\Sigma_{3n}^{3p}| \\
&= (-1)^{3n} (-1)^n |H_n^{p+1}|^2 |\Sigma_n^p| \cdot (-1)^n |\Sigma_n^p|^2 |H_n^{p+1}| \\
&= (-1)^{3n} |H_n^{p+1}|^3 |\Sigma_n^p|^3 \\
&= (A_n^p)^3.
\end{aligned}$$

Identity (C2) (resp. (C3), (C4), (C5), (C6)) are proved in the same manner by using Equalities (L5) and (L11) (resp. (L6) and (L12), (L2) and (L13), (L3) and (L14), (L1) and (L15)) stated in Lemma 5. \square

Proof of Theorems 1-3. Let $p_n = (-1)^n |H_n^1| \cdot |\Sigma_n^0|$ and $q_n = (-1)^n |H_{n+1}^0| \cdot |\Sigma_n^1|$. Then Corollary 6 shows that

$$\begin{aligned}
(1) \quad & p_{3n} = p_n^3, \\
(2) \quad & p_{3n+1} = p_n q_n^2, \\
(3) \quad & p_{3n+2} = p_{n+1} q_n^2, \\
(4) \quad & q_{3n} = p_n^2 q_n, \\
(5) \quad & q_{3n+1} = q_n^3, \\
(6) \quad & q_{3n+2} = p_{n+1}^2 q_n.
\end{aligned}$$

Moreover, the first values are $p_0 = p_1 = p_2 = q_0 = q_1 = q_2 = 1$. By induction, we have $p_n = q_n = 1$ for all $n \geq 0$. In other words,

$$|H_n^1| \cdot |\Sigma_n^0| = (-1)^n \quad \text{and} \quad |H_{n+1}^0| \cdot |\Sigma_n^1| = (-1)^n. \quad (19)$$

The last three identities in (8) are consequences of Equations (L1), (L2), (L3) respectively. Similarly, the last three identities in (9) are consequences of Equations (L4), (L5), (L6) respectively. Thus, Theorem 1 is proved. By Relation (19), Theorem 1 implies Theorem 2. Finally, Theorem 3 is a consequence of Lemma 5, because the set $\{0, \pm 1, \pm J, \pm J^2\}$ is closed under multiplication. \square

3. PROOF OF EQUALITIES (L1)-(L9)

Recall that the Thue–Morse-like sequence $\mathbf{c} = c_0 c_1 \cdots c_n \cdots \in \mathcal{A}^\infty$ is characterized by the recurrent equations in (6), and that $H_n^p := H_n^p(\mathbf{c})$ is the (p, n) -order Hankel matrix of \mathbf{c} . Let $K_n^p := K_n^p(\mathbf{c}) := (c_{p+3(i+j-2)})_{1 \leq i, j \leq n}$. By (6), we have for all $n \geq 1, p \geq 0$,

$$K_n^{3p} = H_n^p, \quad K_n^{3p+1} = J H_n^p, \quad K_n^{3p+2} = \mathbf{0}_{n,n}. \quad (20)$$

Equalities (L1)-(L8) are proved by combining (20) and (16-18) where the sequence \mathbf{u} is specialized to \mathbf{c} . For simplicity, during the proof, we will denote $n+1$ and $p+1$ by \bar{n} and \bar{p} respectively. Also, let $I_{n,n}$ be the identity matrix of size $n \times n$.

(L1) Combine (20) and (16), we have

$$\begin{aligned}
& |H_{3n}^{3p}| = |P^t H_{3n}^{3p} P| \\
&= \begin{vmatrix} H_n^p & JH_n^p & \mathbf{0}_{n,n} \\ JH_n^p & \mathbf{0}_{n,n} & H_n^{\bar{p}} \\ \mathbf{0}_{n,n} & H_n^{\bar{p}} & JH_n^{\bar{p}} \end{vmatrix} \\
&= \begin{vmatrix} \begin{pmatrix} H_n^p & JH_n^p & \mathbf{0}_{n,n} \\ JH_n^p & \mathbf{0}_{n,n} & H_n^{\bar{p}} \\ \mathbf{0}_{n,n} & H_n^{\bar{p}} & JH_n^{\bar{p}} \end{pmatrix} & \begin{pmatrix} I_{n,n} & -JI_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,n} & I_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,n} & -J^2 I_{n,n} & I_{n,n} \end{pmatrix} \end{vmatrix} \\
&= \begin{vmatrix} H_n^p & \mathbf{0}_{n,n} & \mathbf{0}_{n,n} \\ JH_n^p & -J^2 H_n^p - J^2 H_n^{\bar{p}} & H_n^{\bar{p}} \\ \mathbf{0}_{n,n} & \mathbf{0}_{n,n} & JH_n^{\bar{p}} \end{vmatrix} \\
&= (-1)^n |H_n^p| \cdot |H_n^{p+1}| \cdot |\Sigma_n^p|.
\end{aligned}$$

(L2) Combine (20) and (17), we have

$$\begin{aligned}
& |H_{3n+1}^{3p}| = |P^t H_{3n+1}^{3p} P| \\
&= \begin{vmatrix} H_{\bar{n}}^p & (JH_{\bar{n}}^p)^{(\bar{n})} & \mathbf{0}_{\bar{n},n} \\ (JH_{\bar{n}}^p)^{(\bar{n})} & \mathbf{0}_{n,n} & H_n^{\bar{p}} \\ \mathbf{0}_{n,\bar{n}} & H_n^{\bar{p}} & JH_n^{\bar{p}} \end{vmatrix} \\
&= \begin{vmatrix} \begin{pmatrix} H_{\bar{n}}^p & (JH_{\bar{n}}^p)^{(\bar{n})} & \mathbf{0}_{\bar{n},n} \\ (JH_{\bar{n}}^p)^{(\bar{n})} & \mathbf{0}_{n,n} & H_n^{\bar{p}} \\ \mathbf{0}_{n,\bar{n}} & H_n^{\bar{p}} & JH_n^{\bar{p}} \end{pmatrix} & \begin{pmatrix} I_{\bar{n},\bar{n}} & -JI_{\bar{n},n} & \mathbf{0}_{\bar{n},n} \\ \mathbf{0}_{n,\bar{n}} & I_{n,n} & 0 \\ \mathbf{0}_{n,\bar{n}} & -J^2 I_{n,n} & I_{n,n} \end{pmatrix} \end{vmatrix} \\
&= \begin{vmatrix} H_{\bar{n}}^p & \mathbf{0}_{\bar{n},n} & \mathbf{0}_{\bar{n},n} \\ (JH_{\bar{n}}^p)^{(\bar{n})} & -J^2(H_n^p + H_n^{\bar{p}}) & H_n^{\bar{p}} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,n} & JH_n^{\bar{p}} \end{vmatrix} \\
&= (-1)^n |H_n^{p+1}| \cdot |H_{n+1}^p| \cdot |\Sigma_n^p|.
\end{aligned}$$

(L3) Combine (20) and (18), we have

$$\begin{aligned}
& |H_{3n+2}^{3p}| = |P^t H_{3n+2}^{3p} P| \\
&= \begin{vmatrix} H_{\bar{n}}^p & JH_{\bar{n}}^p & \mathbf{0}_{\bar{n},n} \\ JH_{\bar{n}}^p & \mathbf{0}_{\bar{n},\bar{n}} & (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} \\ \mathbf{0}_{n,\bar{n}} & (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \left| \begin{pmatrix} H_{\bar{n}}^p & JH_{\bar{n}}^p & \mathbf{0}_{\bar{n},n} \\ JH_{\bar{n}}^p & \mathbf{0}_{\bar{n},\bar{n}} & (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} \\ \mathbf{0}_{n,\bar{n}} & (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & JH_{\bar{n}}^{\bar{p}} \end{pmatrix} \begin{pmatrix} I_{\bar{n},\bar{n}} & -JI_{\bar{n},\bar{n}} & -J^2(I_{\bar{n},\bar{n}})^{(1)} \\ \mathbf{0}_{\bar{n},\bar{n}} & I_{\bar{n},\bar{n}} & J(I_{\bar{n},\bar{n}})^{(1)} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,\bar{n}} & I_{n,n} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} H_{\bar{n}}^p & \mathbf{0}_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},n} \\ JH_{\bar{n}}^p & -J^2 H_{\bar{n}}^p & \mathbf{0}_{\bar{n},n} \\ \mathbf{0}_{n,\bar{n}} & (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & J(H_{\bar{n}}^{\bar{p}} + H_{\bar{n}}^{\bar{p}+1}) \end{pmatrix} \right| \\
&= (-1)^{n+1} J^2 |H_{n+1}^p|^2 \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

(L4) Combine (20) and (16), we have

$$\begin{aligned}
&|H_{3n}^{3p+1}| = |P^t H_{3n}^{3p+1} P| \\
&= \left| \begin{pmatrix} JH_n^p & \mathbf{0}_{n,n} & H_n^{\bar{p}} \\ \mathbf{0}_{n,n} & H_n^{\bar{p}} & JH_n^{\bar{p}} \\ H_n^{\bar{p}} & JH_n^{\bar{p}} & \mathbf{0}_{n,n} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} JH_n^p & \mathbf{0}_{n,n} & H_n^{\bar{p}} \\ \mathbf{0}_{n,n} & H_n^{\bar{p}} & JH_n^{\bar{p}} \\ H_n^{\bar{p}} & JH_n^{\bar{p}} & \mathbf{0}_{n,n} \end{pmatrix} \begin{pmatrix} I_{n,n} & -JI_{n,n} & J^2 I_{n,n} \\ \mathbf{0}_{n,n} & I_{n,n} & -JI_{n,n} \\ \mathbf{0}_{n,n} & \mathbf{0}_{n,n} & I_{n,n} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} JH_n^p & -J^2 H_n^p & H_n^p + H_n^{\bar{p}} \\ \mathbf{0}_{n \times n} & H_n^{\bar{p}} & \mathbf{0}_{n,n} \\ H_n^{\bar{p}} & \mathbf{0}_{n,n} & \mathbf{0}_{n \times n} \end{pmatrix} \right| \\
&= (-1)^n |H_n^{p+1}|^2 \cdot |\Sigma_n^p|.
\end{aligned}$$

(L5) Combine (20) and (17), we have

$$\begin{aligned}
&|H_{3n+1}^{3p+1}| = |P^t H_{3n+1}^{3p+1} P| \\
&= \left| \begin{pmatrix} JH_{\bar{n}}^p & \mathbf{0}_{\bar{n},n} & (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} \\ \mathbf{0}_{n,\bar{n}} & H_{\bar{n}}^{\bar{p}} & JH_{\bar{n}}^{\bar{p}} \\ (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & JH_{\bar{n}}^{\bar{p}} & \mathbf{0}_{n,n} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} JH_{\bar{n}}^p & \mathbf{0}_{\bar{n},n} & (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} \\ \mathbf{0}_{n,\bar{n}} & H_{\bar{n}}^{\bar{p}} & JH_{\bar{n}}^{\bar{p}} \\ (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & JH_{\bar{n}}^{\bar{p}} & \mathbf{0}_{n,n} \end{pmatrix} \begin{pmatrix} I_{n,n} & \mathbf{0}_{\bar{n},n} & -J^2(I_{\bar{n},\bar{n}})^{(1)} \\ \mathbf{0}_{n,\bar{n}} & I_{n,n} & -JI_{n,n} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,n} & I_{n,n} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} JH_{\bar{n}}^p & \mathbf{0}_{\bar{n},n} & \mathbf{0}_{\bar{n},n} \\ \mathbf{0}_{n,\bar{n}} & H_{\bar{n}}^{\bar{p}} & \mathbf{0}_{n,n} \\ (H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & JH_{\bar{n}}^{\bar{p}} & -J^2(H_{\bar{n}}^{\bar{p}} + H_{\bar{n}}^{\bar{p}+1}) \end{pmatrix} \right| \\
&= (-1)^n J \cdot |H_n^{p+1}| \cdot |H_{n+1}^p| \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

(L6) Combine (20) and (18), we have

$$|H_{3n+2}^{3p+1}| = |P^t H_{3n+2}^{3p+1} P|$$

$$\begin{aligned}
&= \begin{vmatrix} JH_n^p & \mathbf{0}_{\bar{n},\bar{n}} & (H_n^{\bar{p}})^{(\bar{n})} \\ \mathbf{0}_{\bar{n},\bar{n}} & H_n^{\bar{p}} & (JH_n^{\bar{p}})^{(\bar{n})} \\ (H_n^{\bar{p}})^{(\bar{n})} & (JH_n^{\bar{p}})^{(\bar{n})} & \mathbf{0}_{n,n} \end{vmatrix} \\
&= \begin{vmatrix} \begin{pmatrix} JH_n^p & \mathbf{0}_{\bar{n},\bar{n}} & (H_n^{\bar{p}})^{(\bar{n})} \\ \mathbf{0}_{\bar{n},\bar{n}} & H_n^{\bar{p}} & (JH_n^{\bar{p}})^{(\bar{n})} \\ (H_n^{\bar{p}})^{(\bar{n})} & (JH_n^{\bar{p}})^{(\bar{n})} & \mathbf{0}_{n,n} \end{pmatrix} & \begin{pmatrix} I_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},\bar{n}} & -J^2(I_{\bar{n},\bar{n}})^{(1)} \\ \mathbf{0}_{\bar{n},\bar{n}} & I_{\bar{n},\bar{n}} & -J(I_{\bar{n},\bar{n}})^{(\bar{n})} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,\bar{n}} & I_{n,n} \end{pmatrix} \end{vmatrix} \\
&= \begin{vmatrix} JH_n^p & \mathbf{0}_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},n} \\ \mathbf{0}_{\bar{n},\bar{n}} & H_n^{p+1} & \mathbf{0}_{\bar{n},n} \\ (H_n^{\bar{p}})^{(\bar{n})} & (JH_n^{\bar{p}})^{(\bar{n})} & -J^2(H_n^{\bar{p}+1} + H_n^{\bar{p}}) \end{vmatrix} \\
&= (-1)^n J \cdot |H_{n+1}^{p+1}| \cdot |H_{n+1}^p| \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

(L7) Combine (20) and (16), we have

$$\begin{aligned}
&|H_{3n}^{3p+2}| = |P^t H_{3n}^{3p+2} P| \\
&= \begin{vmatrix} \mathbf{0}_{n,n} & H_n^{\bar{p}} & JH_n^{\bar{p}} \\ H_n^{\bar{p}} & JH_n^{\bar{p}} & \mathbf{0}_{n,n} \\ JH_n^{\bar{p}} & \mathbf{0}_{n,n} & H_n^{\bar{p}+1} \end{vmatrix} \\
&= \begin{vmatrix} \begin{pmatrix} \mathbf{0}_{n,n} & H_n^{\bar{p}} & JH_n^{\bar{p}} \\ H_n^{\bar{p}} & JH_n^{\bar{p}} & \mathbf{0}_{n,n} \\ JH_n^{\bar{p}} & \mathbf{0}_{n,n} & H_n^{\bar{p}+1} \end{pmatrix} & \begin{pmatrix} I_{n,n} & \mathbf{0}_{n,n} & \mathbf{0}_{n,n} \\ -J^2 I_{n,n} & I_{n,n} & \mathbf{0}_{n,n} \\ JI_{n,n} & -J^2 I_{n,n} & I_{n,n} \end{pmatrix} \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{0}_{n,n} & \mathbf{0}_{n,n} & JH_n^{\bar{p}} \\ \mathbf{0}_{n,n} & JH_n^{\bar{p}} & \mathbf{0}_{n,n} \\ J(H_n^{\bar{p}} + H_n^{\bar{p}+1}) & -J^2 H_n^{\bar{p}+1} & H_n^{\bar{p}+1} \end{vmatrix} \\
&= (-1)^n |H_n^{p+1}|^2 \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

(L8) Combine (20) and (17), we have

$$\begin{aligned}
&|H_{3n+1}^{3p+2}| = |P^t H_{3n+1}^{3p+2} P| \\
&= \begin{vmatrix} \mathbf{0}_{\bar{n},\bar{n}} & (H_n^{\bar{p}})^{(\bar{n})} & (JH_n^{\bar{p}})^{(\bar{n})} \\ (H_n^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} & \mathbf{0}_{n \times n} \\ (JH_n^{\bar{p}})^{(\bar{n})} & \mathbf{0}_{n \times n} & H_n^{\bar{p}+1} \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{0}_{\bar{n},\bar{n}} & (H_n^{\bar{p}})^{(\bar{n})} & (JH_n^{\bar{p}})^{(\bar{n})} \\ (H_n^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n,\bar{n}} & -J^2 H_n^{\bar{p}} & H_n^{\bar{p}+1} \end{vmatrix} \\
&= 0.
\end{aligned}$$

(L9) Combine (20) and (18), we have

$$|\Sigma_{3n+2}^{3p+2}| = |P^t \Sigma_{3n+2}^{3p+2} P|$$

$$\begin{aligned}
&= \begin{vmatrix} \mathbf{0}_{\bar{n},\bar{n}} & H_{\bar{n}}^{\bar{p}} & (JH_{\bar{n}}^{\bar{p}})^{(\bar{n})} \\ H_{\bar{n}}^{\bar{p}} & JH_{\bar{n}}^{\bar{p}} & \mathbf{0}_{\bar{n},n} \\ (JH_{\bar{n}}^{\bar{p}})^{(\bar{n})} & \mathbf{0}_{n,\bar{n}} & H_n^{p+2} \end{vmatrix} \\
&= \begin{vmatrix} \begin{pmatrix} \mathbf{0}_{\bar{n},\bar{n}} & H_{\bar{n}}^{\bar{p}} & (JH_{\bar{n}}^{\bar{p}})^{(\bar{n})} \\ H_{\bar{n}}^{\bar{p}} & JH_{\bar{n}}^{\bar{p}} & \mathbf{0}_{\bar{n},n} \\ (JH_{\bar{n}}^{\bar{p}})^{(\bar{n})} & \mathbf{0}_{n,\bar{n}} & H_n^{p+2} \end{pmatrix} & \begin{pmatrix} I_{\bar{n},\bar{n}} & -JI_{\bar{n},\bar{n}} & J^2I_{\bar{n},n} \\ \mathbf{0}_{\bar{n},\bar{n}} & I_{\bar{n},\bar{n}} & -JI_{\bar{n},n} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,\bar{n}} & I_{n,n} \end{pmatrix} \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{0}_{\bar{n},\bar{n}} & H_{\bar{n}}^{\bar{p}} & \mathbf{0}_{\bar{n},n} \\ H_{\bar{n}}^{\bar{p}} & \mathbf{0}_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},n} \\ (JH_{\bar{n}}^{\bar{p}})^{(\bar{n})} & -J^2(H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & H_n^{\bar{p}} + H_n^{\bar{p}+1} \end{vmatrix} \\
&= (-1)^{n+1} |H_{n+1}^{p+1}|^2 \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

4. PROOF OF EQUALITIES (L10)-(L18)

Recall that the sequence $\mathbf{s} = s_0 s_1 \cdots s_n \cdots \in \mathcal{A}^\infty$ is characterized by the recurrent equations in (11), and that $\Sigma_n^p := \Sigma_n^p(\mathbf{s})$ (resp. H_n^p) is the (p, n) -order Hankel matrix of the sequence \mathbf{s} (resp. \mathbf{c}). Let $K_n^p := K_n^p(\mathbf{s}) := (s_{p+3(i+j-2)})_{1 \leq i, j \leq n}$. By (11), we have for all $n \geq 1, p \geq 0$,

$$K_n^{3p} = -J^2 H_n^p, \quad K_n^{3p+1} = JH_n^p, \quad K_n^{3p+2} = H_n^{p+1}. \quad (21)$$

Equalities (L10)-(L18) are proved by combining (21) and (16-18) where the sequence \mathbf{u} is specialized to \mathbf{s} .

(L10) Combine (21) and (16), we have

$$\begin{aligned}
&|\Sigma_{3n}^{3p}| = |P^t \Sigma_{3n}^{3p} P| \\
&= \begin{vmatrix} -J^2 H_n^p & JH_n^p & H_n^{\bar{p}} \\ JH_n^p & H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} \\ H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \end{vmatrix} \\
&= \begin{vmatrix} \begin{pmatrix} -J^2 H_n^p & JH_n^p & H_n^{\bar{p}} \\ JH_n^p & H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} \\ H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \end{pmatrix} & \begin{pmatrix} I_{n,n} & J^2 I_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,n} & I_{n,n} & J^2 I_{n,n} \\ \mathbf{0}_{n,n} & \mathbf{0}_{n,n} & I_{n,n} \end{pmatrix} \end{vmatrix} \\
&= \begin{vmatrix} -J^2 H_n^p & \mathbf{0}_{n,n} & H_n^p + H_n^{\bar{p}} \\ JH_n^p & H_n^p + H_n^{\bar{p}} & \mathbf{0}_{n,n} \\ H_n^{\bar{p}} & \mathbf{0}_{n,n} & \mathbf{0}_{n,n} \end{vmatrix} \\
&= (-1)^n |\Sigma_n^p|^2 \cdot |H_n^{p+1}|.
\end{aligned}$$

(L11) Combine (21) and (17), we have

$$|\Sigma_{3n+1}^{3p}| = |P^t \Sigma_{3n+1}^{3p} P|$$

$$\begin{aligned}
&= \begin{vmatrix} -J^2 H_n^p & (JH_n^p)^{(\bar{n})} & (H_n^{\bar{p}})^{(\bar{n})} \\ (JH_n^p)^{(\bar{n})} & H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} \\ (H_n^{\bar{p}})^{(\bar{n})} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \end{vmatrix} \\
&= \begin{vmatrix} -J^2 H_n^p & (JH_n^p)^{(\bar{n})} & (H_n^{\bar{p}})^{(\bar{n})} \\ (JH_n^p)^{(\bar{n})} & H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} \\ (H_n^{\bar{p}})^{(\bar{n})} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \end{vmatrix} \\
&\quad \times \begin{vmatrix} I_{\bar{n},\bar{n}} & J^2(I_{\bar{n},\bar{n}})^{(\bar{n})} & J(I_{\bar{n},\bar{n}})^{(1)} \\ \mathbf{0}_{n,\bar{n}} & I_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,n} & I_{n,n} \end{vmatrix} \\
&= \begin{vmatrix} -J^2 H_n^p & \mathbf{0}_{\bar{n},n} & \mathbf{0}_{\bar{n},n} \\ (JH_n^p)^{(\bar{n})} & H_n^p + H_n^{\bar{p}} & \mathbf{0}_{n,n} \\ (H_n^{\bar{p}})^{(\bar{n})} & \mathbf{0}_{n,n} & J(H_n^{\bar{p}} + H_n^{\bar{p}+1}) \end{vmatrix} \\
&= (-1)^{n+1} J^2 |H_{n+1}^p| \cdot |\Sigma_n^p| \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

(L12) Combine (21) and (18), we have

$$\begin{aligned}
&|\Sigma_{3n+2}^{3p}| = |P^t \Sigma_{3n+2}^{3p} P| \\
&= \begin{vmatrix} -J^2 H_n^p & JH_n^p & (H_n^{\bar{p}})^{(\bar{n})} \\ JH_n^p & H_n^{\bar{p}} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} \\ (H_n^{\bar{p}})^{(\bar{n})} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} \end{vmatrix} \\
&= \begin{vmatrix} -J^2 H_n^p & JH_n^p & (H_n^{\bar{p}})^{(\bar{n})} \\ JH_n^p & H_n^{\bar{p}} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} \\ (H_n^{\bar{p}})^{(\bar{n})} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} \end{vmatrix} \\
&\quad \times \begin{vmatrix} I_{\bar{n},\bar{n}} & J^2 I_{\bar{n},\bar{n}} & J(I_{\bar{n},\bar{n}})^{(1)} \\ \mathbf{0}_{\bar{n},\bar{n}} & I_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},n} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,n+1} & I_{n,n} \end{vmatrix} \\
&= \begin{vmatrix} -J^2 H_n^p & \mathbf{0}_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},n} \\ JH_n^p & H_n^{\bar{p}} & \mathbf{0}_{\bar{n},n} \\ (H_n^{\bar{p}})^{(\bar{n})} & \mathbf{0}_{n,\bar{n}} & J(H_n^{\bar{p}} + H_n^{\bar{p}+1}) \end{vmatrix} \\
&= (-1)^{n+1} J^2 |H_{n+1}^p| \cdot |\Sigma_{n+1}^p| \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

(L13) Combine (21) and (16), we have

$$\begin{aligned}
&|\Sigma_{3n}^{3p+1}| = |P^t \Sigma_{3n}^{3p+1} P| \\
&= \begin{vmatrix} JH_n^p & H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} \\ H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \\ -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} & H_n^{\bar{p}+1} \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \left| \begin{pmatrix} JH_n^p & H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} \\ H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \\ -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} & H_n^{\bar{p}+1} \end{pmatrix} \begin{pmatrix} I_{n,n} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ JI_{n,n} & I_{n,n} & J^2 I_{n,n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & I_{n,n} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} J(H_n^p + H_n^{\bar{p}}) & H_n^{\bar{p}} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & -J^2 H_n^{\bar{p}} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & JH_n^{\bar{p}} & H_n^{\bar{p}} + H_n^{\bar{p}+1} \end{pmatrix} \right| \\
&= (-1)^n |H_n^{p+1}| \cdot |\Sigma_n^p| \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

(L14) Combine (21) and (17), we have

$$\begin{aligned}
&|\Sigma_{3n+1}^{3p+1}| = |P^t \Sigma_{3n+1}^{3p+1} P| \\
&= \left| \begin{pmatrix} JH_n^p & (H_n^{\bar{p}})^{(\bar{n})} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} \\ (H_n^{\bar{p}})^{(\bar{n})} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \\ -J^2 (H_n^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} & H_n^{p+2} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} JH_n^p & (H_n^{\bar{p}})^{(\bar{n})} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} \\ (H_n^{\bar{p}})^{(\bar{n})} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \\ -J^2 (H_n^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} & H_n^{p+2} \end{pmatrix} \right. \\
&\quad \times \left. \begin{pmatrix} I_{n,n} & -J^2 (I_{n,\bar{n}})^{(1)} & J(I_{n,\bar{n}})^{(1)} \\ \mathbf{0}_{n,\bar{n}} & I_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,n} & I_{n,n} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} JH_n^p & \mathbf{0}_{n+1,n} & \mathbf{0}_{\bar{n},n} \\ (H_n^{\bar{p}})^{(\bar{n})} & -J^2 (H_n^{\bar{p}} + H_n^{\bar{p}+1}) & J(H_n^{\bar{p}} + H_n^{\bar{p}+1}) \\ -J^2 (H_n^{\bar{p}})^{(\bar{n})} & J(H_n^{\bar{p}} + H_n^{\bar{p}+1}) & \mathbf{0}_{n,n} \end{pmatrix} \right| \\
&= (-1)^n J |H_{n+1}^p| \cdot |\Sigma_n^{p+1}|^2.
\end{aligned}$$

(L15) Combine (21) and (18), we have

$$\begin{aligned}
&|\Sigma_{3n+2}^{3p+1}| = |P^t \Sigma_{3n+2}^{3p+1} P| \\
&= \left| \begin{pmatrix} JH_n^p & H_n^{\bar{p}} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} \\ H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} & J(H_n^{\bar{p}})^{(\bar{n})} \\ -J^2 (H_n^{\bar{p}})^{(\bar{n})} & J(H_n^{\bar{p}})^{(\bar{n})} & H_n^{\bar{p}+1} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} JH_n^p & H_n^{\bar{p}} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} \\ H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} & J(H_n^{\bar{p}})^{(\bar{n})} \\ -J^2 (H_n^{\bar{p}})^{(\bar{n})} & J(H_n^{\bar{p}})^{(\bar{n})} & H_n^{\bar{p}+1} \end{pmatrix} \right. \\
&\quad \times \left. \begin{pmatrix} I_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},n} \\ JI_{\bar{n},\bar{n}} & I_{\bar{n},\bar{n}} & J^2 (I_{\bar{n},\bar{n}})^{(\bar{n})} \\ \mathbf{0}_{n,\bar{n}} & \mathbf{0}_{n,\bar{n}} & I_{n,n} \end{pmatrix} \right|
\end{aligned}$$

$$\begin{aligned}
&= \begin{vmatrix} J(H_n^{\bar{p}} + H_n^{\bar{p}}) & H_n^{\bar{p}} & \mathbf{0}_{\bar{n},n} \\ \mathbf{0}_{\bar{n},\bar{n}} & -J^2 H_n^{\bar{p}} & \mathbf{0}_{\bar{n},n} \\ \mathbf{0}_{n,\bar{n}} & J(H_n^{\bar{p}})_{(\bar{n})} & H_n^{\bar{p}} + H_n^{\bar{p}+1} \end{vmatrix} \\
&= (-1)^{n+1} |H_{n+1}^{p+1}| \cdot |\Sigma_{n+1}^p| \cdot |\Sigma_n^{p+1}|.
\end{aligned}$$

(L16) Combine (21) and (16), we have

$$\begin{aligned}
&|\Sigma_{3n}^{3p+2}| = |P^t \Sigma_{3n}^{3p+2} P| \\
&= \begin{vmatrix} H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \\ -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} & H_n^{\bar{p}+1} \\ JH_n^{\bar{p}} & H_n^{\bar{p}+1} & -J^2 H_n^{\bar{p}+1} \end{vmatrix} \\
&= \begin{vmatrix} \begin{pmatrix} H_n^{\bar{p}} & -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} \\ -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} & H_n^{\bar{p}+1} \\ JH_n^{\bar{p}} & H_n^{\bar{p}+1} & -J^2 H_n^{\bar{p}+1} \end{pmatrix} & \begin{pmatrix} I_{n,n} & \mathbf{0}_{n,n} & -JI_{n,n} \\ \mathbf{0}_{n,n} & I_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,n} & JI_{n,n} & I_{n,n} \end{pmatrix} \end{vmatrix} \\
&= \begin{vmatrix} H_n^{\bar{p}} & \mathbf{0}_{n,n} & \mathbf{0}_{n,n} \\ -J^2 H_n^{\bar{p}} & JH_n^{\bar{p}} & H_n^{\bar{p}} + H_n^{\bar{p}+1} \\ JH_n^{\bar{p}} & \mathbf{0}_{n,n} & -J^2(H_n^{\bar{p}} + H_n^{\bar{p}+1}) \end{vmatrix} \\
&= (-1)^n |\Sigma_n^{p+1}|^2 \cdot |H_n^{p+1}|.
\end{aligned}$$

(L17) Combine (21) and (17), we have

$$\begin{aligned}
&|\Sigma_{3n+1}^{3p+2}| = |P^t \Sigma_{3n+1}^{3p+2} P| \\
&= \begin{vmatrix} H_n^{\bar{p}} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} & J(H_n^{\bar{p}})^{(\bar{n})} \\ -J^2 (H_n^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} & H_n^{\bar{p}+1} \\ J(H_n^{\bar{p}})^{(\bar{n})} & H_n^{\bar{p}+1} & -J^2 H_n^{\bar{p}+1} \end{vmatrix} \\
&= \begin{vmatrix} \begin{pmatrix} H_n^{\bar{p}} & -J^2 (H_n^{\bar{p}})^{(\bar{n})} & J(H_n^{\bar{p}})^{(\bar{n})} \\ -J^2 (H_n^{\bar{p}})^{(\bar{n})} & JH_n^{\bar{p}} & H_n^{\bar{p}+1} \\ J(H_n^{\bar{p}})^{(\bar{n})} & H_n^{\bar{p}+1} & -J^2 H_n^{\bar{p}+1} \end{pmatrix} & \begin{pmatrix} I_{\bar{n},\bar{n}} & \mathbf{0}_{\bar{n},n} & -J(I_{\bar{n},\bar{n}})^{(\bar{n})} \\ \mathbf{0}_{n,\bar{n}} & I_{n,n} & \mathbf{0}_{n,n} \\ \mathbf{0}_{n,\bar{n}} & JI_{n,n} & I_{n,n} \end{pmatrix} \end{vmatrix} \\
&= \begin{vmatrix} H_n^{\bar{p}} & \mathbf{0}_{\bar{n},n} & \mathbf{0}_{\bar{n},n} \\ -J^2 (H_n^{\bar{p}})^{(\bar{n})} & J(H_n^{\bar{p}} + H_n^{\bar{p}+1}) & H_n^{\bar{p}} + H_n^{\bar{p}+1} \\ J(H_n^{\bar{p}})^{(\bar{n})} & \mathbf{0}_{n,n} & -J^2(H_n^{\bar{p}} + H_n^{\bar{p}+1}) \end{vmatrix} \\
&= (-1)^n |\Sigma_n^{p+1}|^2 \cdot |H_{n+1}^{p+1}|.
\end{aligned}$$

(L18) Combine (21) and (18), we have

$$\begin{aligned}
& |\Sigma_{3n+2}^{3p+2}| = |P^t \Sigma_{3n+2}^{3p+2} P| \\
& = \begin{vmatrix} H_{\bar{n}}^{\bar{p}} & -J^2 H_{\bar{n}}^{\bar{p}} & J(H_{\bar{n}}^{\bar{p}})^{(\bar{n})} \\ -J^2 H_{\bar{n}}^{\bar{p}} & J H_{\bar{n}}^{\bar{p}} & (H_{\bar{n}}^{\bar{p}+1})^{(\bar{n})} \\ J(H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & (H_{\bar{n}}^{\bar{p}+1})^{(\bar{n})} & -J^2 H_{\bar{n}}^{\bar{p}+1} \end{vmatrix} \\
& = \begin{vmatrix} H_{\bar{n}}^{\bar{p}} & \mathbf{0}_{\bar{n}, \bar{n}} & J(H_{\bar{n}}^{\bar{p}})^{(\bar{n})} \\ -J^2 H_{\bar{n}}^{\bar{p}} & \mathbf{0}_{\bar{n}, \bar{n}} & (H_{\bar{n}}^{\bar{p}+1})^{(\bar{n})} \\ J(H_{\bar{n}}^{\bar{p}})^{(\bar{n})} & (\Sigma_{\bar{n}}^{\bar{p}})^{(\bar{n})} & -J^2 H_{\bar{n}}^{\bar{p}+1} \end{vmatrix} \\
& = 0.
\end{aligned}$$

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